11

Indeterminate Forms

Art. 1. Indeterminate forms.

While evaluating limits, we often come across expressions of the form : $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty, \infty^0, 0^0, 1^\infty$ which are all undefined and are called Indeterminate forms.

Art. 2. Indeterminate Forms

Form I. $\frac{0}{0}$ Form.

Theorem 1. Statement. *If f and g are two differentiable functions at $x = a$ and*

(i) $f(a) = 0 = g(a)$

(ii) $g'(a) \neq 0$, then $\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$

Proof. $g'(a) \neq 0$ (given)

$\Rightarrow$ Either $g'(a) > 0$ or $g'(a) < 0$

If $g'(a) > 0$, then $g(x) > g(a) = 0$ (given) for $x > a$

and if $g'(a) < 0$, then $g(x) < g(a) = 0$ for $x > a$

Thus $g'(a) \neq 0 \Rightarrow g(x) > 0$ or $g(x) < 0$ for $x > a$

$\Rightarrow g(x) \neq 0$ in deleted nbd. of a.

Since f and g are derivable at $x = a$ (given)

$\therefore$ f and g are continuous at $x = a$

$\therefore \underset{x \to a}{\text{Lt}} f(x) = f(a) = 0$ and $\underset{x \to a}{\text{Lt}} g(x) = g(a) = 0$

and so $\dfrac{\underset{x \to a}{\text{Lt}} f(x)}{\underset{x \to a}{\text{Lt}} g(x)}$ takes the form $\frac{0}{0}$ *i.e.* indeterminate.

Consider, $\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to a}{\text{Lt}} \frac{f(x) - 0}{g(x) - 0} = \underset{x \to a}{\text{Lt}} \frac{f(x) - f(a)}{g(x) - g(a)}$ $\quad [\because f(a) = 0 = g(a)]$

$$= \underset{x \to a}{\text{Lt}} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} = \frac{\underset{x \to a}{\text{Lt}} \frac{f(x) - f(a)}{x - a}}{\underset{x \to a}{\text{Lt}} \frac{g(x) - g(a)}{x - a}}$$

$$= \frac{f'(a)}{g'(a)}$$ $[\because f$ and g are given to be derivable at $x = a]$

Art. 3. Theorem II. L' Hospital's Rule for 0/0 form. *(G.N.D.U. 1998 Oct.)*

Statement. *If f and g are differentiable in some nbd. N of the point a, except perhaps at a and*

(i) $\underset{x \to a}{Lt} f(x) = 0 = \underset{x \to a}{Lt} g(x)$

(ii) $g'(x) \neq 0$ *in some deleted nbd. N of a*

(*iii*) $\underset{x\to a}{Lt}\dfrac{f'(x)}{g'(x)}$ *exists, whether finite or infinite.*

Then $\underset{x\to a}{Lt}\dfrac{f(x)}{g(x)} = \underset{x\to a}{Lt}\dfrac{f'(x)}{g'(x)}$

Proof. By def. of limit the value of f at $x = a$ is insignificant as $x \to a$, so we may assume that $f(a) = 0 = \underset{x\to a}{Lt} f(x)$ and $g(a) = 0 = \underset{x\to a}{Lt} g(x)$

This assumption makes f and g continuous at $x = a$.

Now f and g satisfy all the conditions of Cauchy's Mean Value Theorem in $[x, a]$ or $[a, x]$ as the case may be. Hence $\exists$ c in (x, a) or (a, x) such that

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(c)} \text{ where } c \in (x,a) or (a,x)$$

$$\Rightarrow \quad \frac{f(x)-0}{g(x)-0} = \frac{f'(c)}{g'(c)} \qquad [\because f(a) = 0 = g(a)]$$

$$\Rightarrow \quad \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \qquad ...(1)$$

where $c \in (x, a)$ or (a, x)

Let $c \in (x, a)$

Now as $x \to a-, c \to a-$ $\qquad [\because x < c < a]$

$$\therefore \quad \underset{x\to a-}{\text{Lt}}\frac{f(x)}{g(x)} = \underset{c\to a-}{\text{Lt}}\frac{f'(c)}{g'(c)} \qquad \text{[Using (1)]}$$

$$= \underset{x\to a-}{\text{Lt}}\frac{f'(x)}{g'(x)} \qquad [\text{Replacing } c \text{ by } x]$$

$$= \underset{x\to a}{\text{Lt}}\frac{f'(x)}{g'(x)} \quad ...(2) \qquad \left[\because \text{By condition } (iii), \ \underset{x\to a}{\text{Lt}}\frac{f'(x)}{g'(x)} \text{ exists}\right]$$

Similarly, when $c \in (a, x)$ *i.e.* $a < c < x$, we can prove

$$\underset{x\to a+}{\text{Lt}}\frac{f(x)}{g(x)} = \underset{x\to a}{\text{Lt}}\frac{f'(x)}{g'(x)} \qquad ...(3)$$

From (2) and (3),

$$\underset{x\to a}{\text{Lt}}\frac{f(x)}{g(x)} \text{ exists and } = \underset{x\to a}{\text{Lt}}\frac{f'(x)}{g'(x)}$$

Note 1. L' Hospital's rule is valid even if $x \to \infty$ or $-\infty$ instead of a.

Here we put $x = \dfrac{1}{t}$ so that

When $x \to \infty, t \to 0+$

$$\therefore \quad \underset{x\to\infty}{\text{Lt}}\frac{f(x)}{g(x)} = \underset{t\to 0+}{\text{Lt}}\frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)} = \underset{t\to 0+}{\text{Lt}}\frac{f'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)}{g'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)} \qquad \text{(By L' Hospital's rule)}$$

$$= \underset{t \to 0+}{\text{Lt}} \frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)} = \underset{x \to \infty}{\text{Lt}} \frac{f'(x)}{g'(x)}$$

Note 2. The theorem holds if we consider any of the one sided limits only.

Note 3. The above theorem is applicable only when all the three conditions of the statement are satisfied by f and g. If any one of the three conditions is not satisfied, then L' Hospital's rule is not applicable.

But this does not mean that

$\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)}$ cannot be evaluated. In this case limit may be evaluated by applying some other method.

Note 4. Condition (*iii*) of theorem *i.e.* $\underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)}$ exists, whether finite or infinite is essential. If this limit does not exist, then $\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)}$ may or may not exist and L' Hospital's rule is not applicable.

Note 5. To simplify the problems, $\underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1 = \underset{x \to 0}{\text{Lt}} \frac{\tan x}{x}$ and $\underset{x \to 0}{\text{Lt}} (1+x)^{1/x} = e$ or any other standard limit should be used before using L' Hospital's Rule, whenever possible.

Note 6. To reduce labour, standard expansions may be used before using L' Hospital's Rule..

Impotant remarks.

If $\underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)}$ is of the form $\frac{0}{0}$ and the functions $f'(x)$ and $g'(x)$ satisfy the conditions of theorem II *i.e.* L' Hospital's rule, then

$$\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)} = \underset{x \to a}{\text{Lt}} \frac{f''(x)}{g''(x)}$$

The rule can be generalized as follows :

➤ **Art. 4. Theorem III. Generalization of L' Hospital's rule.**

If f and g be two functions differentiable upto order n in a deleted nbd. N of a such that

(*i*) $g^{(k)}(x) \neq 0, 0 \leq k \leq n, k \in N$ (*where* $g^0(x) = g(x)$)

(*ii*) $\underset{x \to a}{Lt} f^{(k)}(x) = 0 = \underset{x \to a}{Lt} g^{(k)}(x); 0 \leq k \leq n-1$

(*iii*) $\underset{x \to a}{Lt} \frac{f^n(x)}{g^n(x)}$ *exists, whether finite or infinite, then* $\underset{x \to a}{Lt} \frac{f(x)}{g(x)} = \underset{x \to a}{Lt} \frac{f^n(x)}{g^n(x)}$

Rule to find $\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)} \left(\frac{0}{0} \text{ form}\right) = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)}$

Note. The numerator $f(x)$ and denominator $g(x)$ should be differentiated separately. Do not differentiate $\frac{f(x)}{g(x)}$ by quotient rule.

Solved Examples 11 (a)

Example 1. *Evaluate the following limits :*

(i) $\underset{x\to 0}{Lt}\dfrac{xe^x-\log(1+x)}{x^2}$ (G.N.D.U. 1993) (ii) $\underset{x\to 0}{Lt}\dfrac{1-\cos^2 x}{\sin x^2}$

(iii) $\underset{x\to 0}{Lt}\dfrac{(\tan^{-1}x)^2}{\log(1+x^2)}$ (iv) $\underset{x\to\infty}{Lt}\dfrac{\tan^{-1}\left(\dfrac{2}{x}\right)}{\dfrac{1}{x}}$

(P.U.2003, M.D.U 2001, K.U

(v) $\underset{x\to 0}{Lt}\dfrac{\cosh x-\cos x}{x\sin x}$ (G.N.D.U

(vi) $\underset{x\to\infty}{Lt}\, x\sin\dfrac{1}{x}$

Solution. (i) $\underset{x\to 0}{\text{Lt}}\dfrac{xe^x-\log(1+x)}{x^2}\left(\dfrac{0}{0}\text{form}\right)$ (Use L' Hospital's rule)

$$=\underset{x\to 0}{\text{Lt}}\frac{xe^x+e^x-\left(\dfrac{1}{1+x}\right)}{2x}\left(\frac{0}{0}\text{form}\right) \text{ (Use L' Hospital's rule)}$$

$$=\underset{x\to 0}{\text{Lt}}\frac{xe^x+2e^x+\dfrac{1}{(1+x)^2}}{2}=\frac{0+2+1}{2}+\frac{3}{2}$$

(ii) $\underset{x\to 0}{\text{Lt}}\dfrac{1-\cos^2 x}{\sin x^2}=\underset{x\to 0}{\text{Lt}}\dfrac{\sin^2 x}{\sin x^2}\left(\dfrac{0}{0}\text{form}\right)$

$$=\underset{x\to 0}{\text{Lt}}\frac{2\sin x\cos x}{2x\cos x^2} \text{ (By L' Hospital's rule)}$$

$$=\left(\underset{x\to 0}{\text{Lt}}\frac{\sin x}{x}\right)\left(\underset{x\to 0}{\text{Lt}}\frac{\cos x}{\cos x^2}\right)=(1)\left(\frac{1}{1}\right)=1.$$

(iii) $\underset{x\to 0}{\text{Lt}}\dfrac{(\tan^{-1}x)^2}{\log(1+x^2)}\left(\dfrac{0}{0}\text{form}\right)$ (Use L' Hospital's rule)

$$=\underset{x\to 0}{\text{Lt}}\frac{2\tan^{-1}x\left(\dfrac{1}{1+x^2}\right)}{\left(\dfrac{1}{1+x^2}\right)(2x)}=\underset{x\to 0}{\text{Lt}}\frac{\tan^{-1}x}{x}\left(\frac{0}{0}\text{ form}\right)=\underset{x\to 0}{\text{Lt}}\frac{\left(\dfrac{1}{1+x^2}\right)}{1}=1$$

(iv) $$\underset{x\to\infty}{\text{Lt}}\frac{\tan^{-1}\left(\dfrac{2}{x}\right)}{\dfrac{1}{x}}\left(\frac{0}{0}\text{form}\right)=\underset{x\to\infty}{\text{Lt}}\frac{\left[\dfrac{1}{1+\dfrac{4}{x^2}}\right]\left(-\dfrac{2}{x^2}\right)}{\left(-\dfrac{1}{x^2}\right)}=\underset{n\to\infty}{\text{Lt}}\frac{2}{1+\dfrac{4}{x^2}}=\frac{2}{1+0}=$$

(v) $\underset{x\to 0}{\mathrm{Lt}}\ \dfrac{\cosh x-\cos x}{x\sin x}=\underset{x\to 0}{\mathrm{Lt}}\ \dfrac{\cosh x-\cos x}{x^2}\left(\dfrac{x}{\sin x}\right)$

$$=\underset{x\to 0}{\mathrm{Lt}}\ \frac{\cosh x-\cos x}{x^2}\left(\frac{0}{0}\text{ form}\right) \qquad \left[\underset{x\to 0}{\mathrm{Lt}}\ \frac{x}{\sin x}=1\right]$$

$$=\underset{x\to 0}{\mathrm{Lt}}\ \frac{\sinh x+\sin x}{2\,.\,x}\left(\text{which is }\frac{0}{0}\text{ form}\right) \quad \text{(By L' Hospital's rule)}$$

$$=\underset{x\to 0}{\mathrm{Lt}}\ \frac{\cosh x+\cos x}{2}=\frac{1+1}{2}=\frac{2}{2}=1$$

(vi) $\underset{x\to\infty}{\mathrm{Lt}}\ x\sin\dfrac{1}{x}=\underset{x\to\infty}{\mathrm{Lt}}\ \dfrac{\sin 1/x}{1/x}\left(\dfrac{0}{0}\text{ form}\right)=\underset{x\to\infty}{\mathrm{Lt}}\ \dfrac{\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{\left(-1/x^2\right)}$

$$=\underset{x\to\infty}{\mathrm{Lt}}\ \cos\frac{1}{x}=\cos 0=1$$

Example 2. *Determine the values of a and b for which*

$$\underset{x\to 0}{\mathrm{Lt}}\ \frac{x(1+a\cos x)-b\sin x}{x^3}=1$$

(Pbi.U.2000 ; P.U.2000, 1998 Oct ; K.U.1996 ; G.N.D.U. 2003,1995 ; H.P.U.1998)

Solution. $\underset{x\to 0}{\mathrm{Lt}}\ \dfrac{x(1+a\cos x)-b\sin x}{x^3}\left(\dfrac{0}{0}\text{ form}\right)$

$$=\underset{x\to 0}{\mathrm{Lt}}\ \frac{x(-a\sin x)+(1+a\cos x)-b\cos x}{3x^2}$$

$$=\underset{x\to 0}{\mathrm{Lt}}\ \frac{1+(a-b)\cos x-ax\sin x}{3x^2}$$

$$\left[\begin{array}{l}\text{Form }\dfrac{1+a-b}{0}.\ \text{Since limit}=1,\text{ it must assume the form }\dfrac{0}{0}\\ \text{as otherwise limit becomes infinite and for this }1+a-b=0\end{array}\right]$$

Form $\dfrac{0}{0}$ for $1+a-b=0$ ...(1)

$$=\underset{x\to 0}{\mathrm{Lt}}\ \frac{(a-b)(-\sin x)-a(x\cos x+\sin x)}{6x}$$

$$=\underset{x\to 0}{\mathrm{Lt}}\ \frac{(2a-b)(-\sin x)-ax\cos x}{6x}\left(\frac{0}{0}\text{ Form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{(2a-b)(-\cos x) - a(-x\sin x + \cos x)}{6}$$

$$= \frac{(2a-b)(-1)-a}{6} = \frac{-3a+b}{6}$$

But limit is given to be $= 1$

Hence $\quad \dfrac{-3a+b}{6} = 1$

$\Rightarrow \quad -3a + b = 6$

Also $\quad 1 + a - b = 0$ ...(2)

From (1) and (2) on solving, we have

$$a = -\frac{5}{2}, b = -\frac{3}{2}$$

Note. In H.P.U. 1998 paper x is changed to θ and the problem is, find a and b so that

$$\underset{\theta \to 0}{\text{Lt}} \frac{\theta(1+a\cos\theta) - b\sin\theta}{\theta^3} = 1 \qquad \textit{(H.P.U. 1998)}$$

Example 3. *Criticise the following or what is the fallacy in*

$$\underset{x\to 1}{Lt}\left(\frac{x^2+x-2}{5x^3-5x^2+x-1}\right) = \underset{x\to 1}{Lt}\left(\frac{2x+1}{15x^2-10x+1}\right)$$

$$= \underset{x\to 1}{Lt}\left(\frac{2}{30x-10}\right) = \underset{x\to 1}{Lt}\left(\frac{0}{30}\right) = 0$$

Solution. First step is correct

i.e. $\underset{x\to 1}{\text{Lt}}\left(\dfrac{x^2+x-2}{5x^3-5x^2+x-1}\right)\left[\dfrac{0}{0}\text{form}\right] = \underset{x\to 1}{\text{Lt}}\left(\dfrac{2x+1}{15x^2-10x+1}\right)$

Now $\underset{x\to 1}{\text{Lt}}\left(\dfrac{2x+1}{15x^2-10x+1}\right)$ is not of the form $\dfrac{0}{0}$, so further application of L' Hospital's rule is absurd and not valid.

The correct solution after first step is :

$$\underset{x\to 1}{\text{Lt}}\left(\frac{2x+1}{15x^2-10x+1}\right) = \frac{2(1)+1}{15(1)^2-10(1)+1} = \frac{3}{6} = \frac{1}{2}$$

Example 4. *Show that* $\underset{x\to 0}{Lt} \dfrac{x^2 \sin 1/x}{\tan x}$ *exists, but cannot be evaluated by L'Hospital's rule. What is the limit ?* *(H.P.U.1996 ; P.U. 1999)*

Solution. Let $f(x) = x^2 \sin \dfrac{1}{x} \Rightarrow \underset{x\to 0}{\text{Lt}} f(x)$

$$= \underset{x\to 0}{\text{Lt}}\, x^2 \sin\frac{1}{x} = 0 \qquad \left[\because \underset{x\to 0}{\text{Lt}}\, x^2 = 0 \text{ and } \left|\sin\frac{1}{x}\right| \le 1 \,\forall x \text{ in nbd. of } 0\right]$$

and $g(x) = \tan x \Rightarrow \underset{x\to 0}{\text{Lt}}\, g(x) = \underset{x\to 0}{\text{Lt}}\, \tan x = 0$

So, $\underset{x\to 0}{\text{Lt}}\, f(x) = 0 = \underset{x\to 0}{\text{Lt}}\, g(x)$ *and therefore condition (i) of L' Hospital's rule is satisfied.*

Also $g'(x) = \sec^2 x \neq 0$ for any x in nbd of 0 and *therfore condition (ii) of L'Hospital's rule is also satisfied.*

Again $$\underset{x\to 0}{\text{Lt}}\frac{f'(x)}{g'(x)} = \underset{x\to 0}{\text{Lt}}\frac{\frac{d}{dx}\left(x^2\sin\frac{1}{x}\right)}{\frac{d}{dx}(\tan x)}$$

$$= \underset{x\to 0}{\text{Lt}}\frac{x^2\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)+\left(\sin\frac{1}{x}\right)(2x)}{\sec^2 x} = \underset{x\to 0}{\text{Lt}}\frac{-\cos\frac{1}{x}+2\left(x\sin\frac{1}{x}\right)}{\sec^2 x}$$

and this limit does not exist because $\underset{x\to 0}{\text{Lt}}\cos\frac{1}{x}$ does not exist.

Therefore, *condition (iii)* of *L'Hospital's rule is not satisfied.*

Hence, L' Hospital's rule can't be applied to evaluate the given limit.

However, the limit exists as shown below

$$\underset{x\to 0}{\text{Lt}}\frac{x^2\sin 1/x}{\tan x} = \underset{x\to 0}{\text{Lt}}\left(\frac{x}{\tan x}\right)\left(x\sin\frac{1}{x}\right) = \underset{x\to 0}{\text{Lt}}\left(\frac{x}{\tan x}\right)\left(\underset{x\to 0}{\text{Lt}}\, x\sin\frac{1}{x}\right)$$

$$= (1)\,(0) \quad \left[\begin{array}{l}\because \underset{x\to 0}{\text{Lt}}\frac{x}{\tan x}=1; \left|\sin\frac{1}{x}\right| \le 1 \;\forall\; x \text{ in} \\ \text{nbd of 0 and } \underset{x\to 0}{\text{Lt}}\, x = 0 \Rightarrow \underset{x\to 0}{\text{Lt}}\, x\sin\frac{1}{x} = 0\end{array}\right]$$

$$= 0.$$

Exercise 11(a)

Evaluate the follwing limits :

1. (i) $\underset{x\to 0}{\text{Lt}}\frac{\sin ax}{\sin bx}$ (ii) $\underset{x\to 0}{\text{Lt}}\frac{x-\tan^{-1}x}{x-\sin x}$ (iii) $\underset{x\to\frac{\pi}{2}}{\text{Lt}}\frac{\cos x}{\frac{\pi}{2}-x}$

(iv) $\underset{x\to 0}{\text{Lt}}\frac{x-\tan x}{x-\sin x}$ (v) $\underset{x\to 0}{\text{Lt}}\frac{(1+\sin x)^{1/3}-(1-\sin x)^{1/3}}{x}$

(G.N.D.U. 1998 ; Pbi.U.2002)

(vi) $\underset{x\to 0}{\text{Lt}}\frac{\cot x-\frac{1}{x}}{x}$

2. (i) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$ (ii) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x - e^{-x} - 2\log(1+x)}{x\sin x}$

(K.U.1996, Pbi.U.1998 ; H.P.U. 2001 S)

(iii) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$ (iv) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x - e^{\sin x}}{x - \sin x}$

(K.U.1998)

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(v) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x + e^{-x} - 2\cos x}{x\sin x}$

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(M.D.U.2001, K.U.2000)

3. (i) $\underset{x\to 0}{\text{Lt}} \dfrac{\sin\log(1+x)}{\log(1+\sin x)}$ (ii) $\underset{x\to 1}{\text{Lt}} (1-x)\cot\dfrac{\pi x}{2}$

(iii) $\underset{x\to 0}{\text{Lt}} \dfrac{\log(1-x^2)}{\log\cos x}$ (iv) $\underset{x\to 0+}{\text{Lt}} \dfrac{3^x - 2^x}{\sqrt{x}}$

(v) $\underset{x\to 0}{\text{Lt}} \dfrac{x\cos x - \log(x+1)}{x^2}$ (vi) $\underset{x\to 1}{\text{Lt}} \dfrac{x^x - x}{1 - x + \log x}$

(M.D.U. 2001)

(vii) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x + e^{-x} + 2\cos x - 4}{x^4}$ (viii) $\underset{x\to 0}{\text{Lt}} \dfrac{x - \sin x}{x^3}$

(ix) $\underset{x\to 0}{\text{Lt}} \dfrac{x - \tan x}{x^3}$

4. (i) Find a and b if $\underset{x\to 0}{\text{Lt}} \dfrac{x(1 - a\cos x) + b\sin x}{x^3}$ exists and equals $\dfrac{1}{3}$

(ii) Given $\underset{x\to 0}{\text{Lt}} \dfrac{\sin x + ax + bx^3}{x^5}$ is finite. Find a and b. Find value of limit also.

(P.U.1990 S)

(iii) Find value of a, b, c, if $\underset{x\to 0}{\text{Lt}} \dfrac{(a + b\cos x)x - c\sin x}{x^5} = 1$ (G.N.D.U.1996)

(iv) Find value of a, b, c if $\underset{x\to 0}{\text{Lt}} \dfrac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$

(M.D.U. 2004, D.L.U. 2004, K.U. 1997, P.U.2001 S, 2002, 1998)

(v) If $\underset{x\to 0}{\text{Lt}} \dfrac{\sin 2x + k\sin x}{x^3}$ be finite ; find value of k and limit also.

(M.D.U. 1997, H.P.U. 2001 ; G.N.D.U. 1999)

(vi) Find a, b and c so that $\underset{x\to 0}{\text{Lt}} \dfrac{a\sin x - bx + cx^2 + x^3}{2x^2\log(1+x) - 2x^3 + x^4}$ is finite. Also determine the limit.

(Pbi. U. 2001 ; G.N.D.U.1997 ; P.U. 2001, 2000 S) (*vii*) Find a and b so that $\underset{x\to 0}{\text{Lt}} \frac{x(1+a\cosh x)-b\sinh x}{x^3}=1$ (*P.U.1995*)

(*viii*) Find p, q, r, if $\underset{y\to 0}{\text{Lt}} \frac{re^y - q\cos y + pe^{-y}}{y\tan y}=3$ (*M.D.U. 1994, K.U. 2004*)

5. What is the fallacy in the following limit while using L'Hospital's law

(*i*) $\underset{x\to 1}{\text{Lt}} \frac{x^3+3x-4}{2x^2+x-3} = \underset{x\to 1}{\text{Lt}} \frac{3x^2+3}{4x+1} = \underset{x\to 1}{\text{Lt}} \frac{6x}{4} = \frac{3}{2}$

(*ii*) $\underset{x\to 1}{\text{Lt}} \frac{x^4-4x^3+3}{3x^2-x-2} = \underset{x\to 1}{\text{Lt}} \frac{4x^3-12x^2}{6x-1} = \underset{x\to 1}{\text{Lt}} \frac{12x^2-24x}{6} = -2$

Find value of correct limit also.

6. Show that the following limits exist but L' Hospital's rule is not applicable. Evaluate the limits also.

(*i*) $\underset{x\to 0}{\text{Lt}} \frac{\sin x^2 \cos\frac{1}{x}}{x}$ (*P.U.2000 S*)

(*ii*) $\underset{x\to 0}{\text{Lt}} \frac{x^2 \sin\frac{1}{x}}{\sin x}$ (*H.P.U.1995 ; 1997*)

Answers

$\text{Log}\,\infty = \infty$

$\text{Log}\,0 = -\infty$

1. (*i*) $\frac{a}{b}$ (*ii*) 2 (*iii*) 1 (*iv*) -2 (*v*) $\frac{2}{3}$ (*vi*) $\frac{-1}{3}$
2. (*i*) 2 (*ii*) 1 (*iii*) $\frac{-2}{3}$ (*iv*) 1 (*v*) 2
3. (*i*) 1 (*ii*) $\frac{-2}{\pi}$ (*iii*) 2 (*iv*) 0 (*v*) $\frac{1}{2}$ (*vi*) -2 (*vii*) $\frac{1}{6}$ (*viii*) $\frac{1}{6}$ (*ix*) $\frac{-1}{3}$
4. (*i*) $a=\frac{1}{2}, b=\frac{-1}{2}$ (*ii*) $a=-1, b=\frac{1}{6}$, Limit $=\frac{1}{120}$

   (*iii*) $a = 120, b = 60, c = 180$ (*iv*) $a = 1, b = 2, c = 1$

   (*v*) K $= -2$; Limit $= -1$ (*vi*) $a = 6, b = 6, c = 0$, Limit $= (3/40)$

   (*vii*) a = 7/2, b = 9/2 (*viii*) $p = \frac{3}{2}, q = 3, r = \frac{3}{2}$
5. (*i*) $\frac{6}{5}$ (*ii*) $\frac{-8}{5}$
6. (*i*) 0 (*ii*) 0

Art. 5. Form II. The indeterminate form $\frac{\infty}{\infty}$

Theorem. *L' Hospital's rule for $\frac{\infty}{\infty}$ form*

Statement. Let f and g be two functions defined and differentiable in deleted nbd N of a such that

(*i*) $\underset{x\to a}{\text{Lt}} f(x) = \infty = \underset{x\to a}{\text{Lt}}\, g(x)$

(*ii*) $g'(x) \neq 0$ for all x in N

(iii) $\underset{x \to a}{\text{Lt}} \dfrac{f'(x)}{g'(x)}$ exists, then $\underset{x \to a}{\text{Lt}} \dfrac{f(x)}{g(x)} = \underset{x \to a}{\text{Lt}} \dfrac{f'(x)}{g'(x)}$

Proof. Let N $= (a-\delta, a+\delta)$ except a. Consider two points x and b satisfying :

$$a < x < b < a+\delta \text{ or } a-\delta < b < x < a$$

These inequalities define the intervals $[x, b]$ or $[b, x]$.

Now f and g satisfy all the conditions of Cauchy's mean value theorem in $[x, b]$. So, $\exists$ c between x and b such that

$$\frac{f(b)-f(x)}{g(b)-g(x)} = \frac{f'(c)}{g'(c)}, x < c < b$$

$$\Rightarrow \quad \frac{f(x)}{g(x)} \left\{ \begin{array}{c} \dfrac{f(b)}{f(x)} - 1 \\ \hline \dfrac{g(b)}{g(x)} - 1 \end{array} \right\} = \frac{f'(c)}{g'(c)} \quad \text{...(1)}$$

Given $\underset{x \to a}{\text{Lt}} f(x) = \infty \Rightarrow \underset{x \to a}{\text{Lt}} \dfrac{f(b)}{f(x)} = 0$ ...(2)

and $\underset{x \to a}{\text{Lt}} g(x) = \infty \Rightarrow \underset{x \to a}{\text{Lt}} \dfrac{g(b)}{g(x)} = 0$ ...(3)

$\infty = \infty$ log

$0 = 0$ fol

Using (2) and (3) in (1), we have

$$\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{c \to a}{\text{Lt}} \frac{f'(c)}{g'(c)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)} \qquad [\because c \to a \text{ as } x \to a]$$

Note. This theorem holds for one sided limits and as $x \to \infty$ or $x \to -\infty$.

If $\dfrac{f'(x)}{g'(x)}$ is again of the form $\dfrac{\infty}{\infty}$ or $\dfrac{0}{0}$, we apply L' Hospital's rule and repeat the process till the limit is finally evaluated.

Solved Examples 11 (b)

Example 1. *Evaluate the follwing limits*

(i) $\underset{x \to 0}{Lt} \dfrac{\log x^2}{\cot x^2}$ *(H.P.U.2000)* *(ii)* $\underset{\theta \to \frac{\pi}{2}+}{Lt} \dfrac{\log\left(\theta - \frac{\pi}{2}\right)}{\tan\theta}$ *(iii)* $\underset{x \to \infty}{Lt} \dfrac{x^2+2x}{5-3x^2}$

Solution. *(i)* $\underset{x \to 0}{\text{Lt}} \dfrac{\log x^2}{\cot x^2} \left(\dfrac{\infty}{\infty} \text{ form}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{x^2}(2x)}{\left\{-\operatorname{cosec}^2\left(x^2\right)\right\}(2x)} = \underset{x \to 0}{\text{Lt}} \frac{-\sin^2\left(x^2\right)}{x^2} \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{(-2)\left(\sin x^2\right)\left(\cos x^2\right)(2x)}{(2x)}$$

$$= \underset{x\to 0}{\text{Lt}}(-2)\left(\sin x^2\right)\left(\cos x^2\right) = (-2)(0)(1) = 0$$

(ii) $$\underset{\theta\to\frac{\pi}{2}+}{\text{Lt}} \frac{\log\left(\theta-\frac{\pi}{2}\right)}{\tan\theta}\left(\frac{\infty}{\infty}\text{form}\right) = \underset{\theta\to\frac{\pi}{2}+}{\text{Lt}} \frac{\left(\frac{1}{\theta-\pi/2}\right)}{\sec^2\theta} = \underset{\theta\to\frac{\pi}{2}+}{\text{Lt}} \frac{\cos^2\theta}{\theta-\frac{\pi}{2}}\left(\frac{0}{0}\text{form}\right)$$

$$= \underset{\theta\to\frac{\pi}{2}+}{\text{Lt}} \frac{2\cos\theta(-\sin\theta)}{1} = 2\,(0)\,(-1) = 0$$

(iii) $$\underset{x\to\infty}{\text{Lt}} \frac{x^2+2x}{5-3x^2}\left(\frac{\infty}{\infty}\text{form}\right) = \underset{x\to\infty}{\text{Lt}} \frac{2x+2}{-6x}\left(\frac{\infty}{\infty}\text{form}\right) = \underset{x\to\infty}{\text{Lt}} \frac{2}{-6} = \frac{-1}{3}$$

Example 2. *Prove that* $\underset{x\to\infty}{Lt} \frac{x^2-\cos x^2}{x^2}$ *exists, but it cannot be evaluated by L'Hospital's rule. Evaluate this limit by some other method.*

Or

What is the fallacy in the following argument ?

$$\underset{x\to\infty}{Lt} \frac{x^2-\cos x^2}{x^2} = \underset{x\to\infty}{Lt} \frac{2x+2x\sin x^2}{2x} = \underset{x\to\infty}{Lt}\left(1+\sin x^2\right)$$

doses not exist and so the given limit does not exist.

Solution. Let $f(x) = x^2 - \cos x^2 \Rightarrow \underset{x\to\infty}{\text{Lt}} f(x) = \infty$

and $g(x) = x^2 \Rightarrow \underset{x\to\infty}{\text{Lt}} g(x) = \infty$

So, condition (*i*) of L' Hospital's rule

i.e. $\underset{x\to\infty}{\text{Lt}} f(x) = \infty = \underset{x\to\infty}{\text{Lt}} g(x)$ is satisfied.

Also, $g'(x) = 2x \neq 0 \;\forall x$ in nbd of ∞

i.e. **Condition (*ii*) of L' Hospital's rule is also satisfied.**

Now $$\underset{x\to\infty}{\text{Lt}} \frac{f'(x)}{g'(x)} = \underset{x\to\infty}{\text{Lt}} \frac{2x+\left(\sin x^2\right)2x}{2x} = \underset{x\to\infty}{\text{Lt}}\left(1+\sin x^2\right)$$

does not exist because $\underset{x\to\infty}{\text{Lt}} \sin x^2$ does not exist, it occillates between -1 and 1.

∴ condition (*iii*) of L' Hospital's rule is not satisfied.
Hence, L'Hospital's rule is not applicable.
However, the given limit exists as shown below :

$$\underset{x\to\infty}{\text{Lt}} \frac{x^2-\cos x^2}{x^2} = \underset{x\to\infty}{\text{Lt}}\left(1-\frac{\cos x^2}{x^2}\right) = \underset{x\to\infty}{\text{Lt}}\, 1 - \underset{x\to\infty}{\text{Lt}} \frac{\cos x^2}{x^2}$$

$$= 1 - \underset{x\to 0+}{\text{Lt}} \frac{\cos\left(\frac{1}{x}\right)^2}{\left(\frac{1}{x}\right)^2} = 1 - \underset{x\to 0+}{\text{Lt}}\, x^2.\cos\frac{1}{x^2}$$

$= 1 - 0 \qquad \left[\because \underset{x\to 0+}{\text{Lt}} x^2 = 0 \text{ and } \cos\frac{1}{x^2} \text{ is bounded in right deleted nbd of } 0\right]$

$= 1$

Exercise 11(b)

1. Evaluate the following limits

(i) $\underset{x\to\frac{\pi}{2}-}{\text{Lt}} \frac{\tan x}{\log\cos x}$ (ii) $\underset{x\to\frac{\pi}{2}}{\text{Lt}} \frac{\tan x}{\tan 3x}$ (iii) $\underset{x\to 0+}{\text{Lt}} \frac{\log\sin x}{\cot x}$ (Pbi. U. 2001)

2. Evaluate the following limits

(i) $\underset{x\to 1-}{\text{Lt}} \frac{\log(1-x)}{\cot\pi x}$ (M.D.U. 1997) (ii) $\underset{x\to 1-}{\text{Lt}} (1-x)\cot\frac{\pi x}{2}$ (iii) $\underset{x\to\frac{\pi}{2}+}{\text{Lt}} \frac{\log\left(x-\frac{\pi}{2}\right)}{\tan x}$

3. Evaluate the following limits

(i) $\underset{x\to 0+}{\text{Lt}} \log_{\tan x}\tan 2x$ (ii) $\underset{x\to 0+}{\text{Lt}} \log_{\sin 2x}\sin x$ (P.U. 2000 S; G.N.D.U. 1999 April)

4. Evaluate the following limits

(i) $\underset{x\to\infty}{\text{Lt}} \frac{\log x}{x}$ (ii) $\underset{x\to\infty}{\text{Lt}} \frac{(\log x)^n}{x}, n\in\mathbb{N}$ (iii) $\underset{x\to c+}{\text{Lt}} \frac{\log(x-c)}{\log\left(e^x-e^c\right)}$

(iv) $\underset{x\to\infty}{\text{Lt}} \frac{e^x-e^{-x}}{e^x+e^{-x}} = 1$ (v) $\underset{x\to\infty}{\text{Lt}} \frac{x^n}{e^x}, n\in\mathbb{N}$ (P.U. 1999 S) (vi) $\underset{x\to 0}{\text{Lt}} \frac{\log x^2}{\cot x^2}$ (H.P.U. 2000)

5. Prove that $\underset{x\to\infty}{\text{Lt}} \frac{x-\sin x}{x}$ exists but it cannot be evaluated by L' Hospital's rule. Evaluate it by some other method.

Answers

1. (i) $-\infty$ (ii) 3 (iii) 0 2. (i) 0 (ii) 0 (iii) 0 3. (i) 1 (ii) 1
4. (i) 0 (ii) 0 (iii) 1 (iv) 1 (v) 0 (vi) 0 5. 1

Art. 6. Form III and IV.

The indeterminate forms $0.\infty$ and $\infty-\infty$

(i) If $\underset{x\to a}{\text{Lt}} f(x) = 0$ and $\underset{x\to a}{\text{Lt}} g(x) = \infty$, then

$$\underset{x\to a}{\text{Lt}} \{f(x).g(x)\} (\text{Form } 0.\infty) = \underset{x\to a}{\text{Lt}} \frac{f(x)}{1/g(x)} \left(\text{Form} \frac{0}{0}\right)$$

or $$\underset{x\to a}{\text{Lt}} \frac{g(x)}{1/f(x)} \left(\text{Form} \frac{\infty}{\infty}\right)$$

and hence can be evaluated by L'Hospital's rule.

(ii) If $\underset{x\to a}{\text{Lt}} f(x) = \infty$ and $\underset{x\to a}{\text{Lt}} g(x) = \infty$, then

$$\underset{x\to a}{\text{Lt}} \{f(x) - g(x)\} (\text{Form } \infty - \infty)$$

$$= \underset{x \to a}{\text{Lt}} \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)g(x)}} \left(\text{Form } \frac{0}{0}\right)$$

and hence can be evaluated by L'Hospital's rule.

Solved Examples 11 (c)

Example 1. *Evaluate the following limits :*

(i) $\underset{x \to 1}{Lt} (1-x) \tan \frac{\pi x}{2}$ (ii) $\underset{x \to 0}{Lt} x^2 \log x^2$

(iii) $\underset{x \to 0+}{Lt} x^m (\log x)^n$, *where m and n* $\in N$ *(P.U.2000)*

Solution. $\underset{x \to 1}{\text{Lt}} (1-x) \tan \frac{\pi x}{2} (\text{Form } 0.\infty) = \underset{x \to 1}{\text{Lt}} \frac{1-x}{\cot \frac{\pi x}{2}} \left(\text{Form } \frac{0}{0}\right)$

$$= \underset{x \to 1}{\text{Lt}} \frac{-1}{-\text{cosec}^2 \left(\frac{\pi x}{2}\right) \frac{\pi}{2}} \quad \text{(Using L'Hospital's Rule)}$$

$$= \underset{x \to 1}{\text{Lt}} \frac{2}{\pi} \sin^2 \frac{\pi x}{2} = \left(\frac{2}{\pi}\right)(1) = \frac{2}{\pi}$$

(ii) $\underset{x \to 0}{\text{Lt}} x^2 \log x^2 \; (\text{Form } 0.\infty)$

$$= \underset{x \to 0}{\text{Lt}} \frac{\log x^2}{1/x^2} \left(\text{Form } \frac{\infty}{\infty}\right) = \underset{x \to 0}{\text{Lt}} \frac{\left(\frac{1}{x^2}\right)(2x)}{-2/x^3} = \underset{x \to 0}{\text{Lt}} \left(-x^2\right) = 0$$

Note. In the indeterminate form $0.\infty$, if $\underset{x \to 0}{\text{Lt}} f(x).g(x)$ involves $\log x$, then take $\log x$ in the numerator such that the form reduces to $\frac{\infty}{\infty}$.

(iii) $\underset{x \to 0+}{\text{Lt}} x^m (\log x)^n$, where m and $n \in \text{N}$ *i.e.* m, n are +ve integers $(\text{Form } 0.\infty)$

$$= \underset{x \to 0+}{\text{Lt}} \frac{(\log x)^n}{x^{-m}} \left(\text{Form } \frac{\infty}{\infty}\right)$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{n(\log x)^{n-1} . 1/x}{-m\left(x^{-m-1}\right)} = \underset{x \to 0+}{\text{Lt}} \frac{n(\log x)^{n-1}}{(-m)\left(x^{-m}\right)} \left(\text{Form } \frac{\infty}{\infty}\right)$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{n(n-1)(\log x)^{n-2}\left(\frac{1}{x}\right)}{(-m)(-m)\left(x^{-m-1}\right)} = \underset{x \to 0+}{\text{Lt}} \frac{n(n-1)(\log x)^{n-2}}{(-m)^2 . x^{-m}} \left(\text{Form } \frac{\infty}{\infty}\right)$$

(We have applied L'Hospital's Rule two times, continue applying n times)

$$= \underset{x\to 0+}{\text{Lt}} \frac{n(n-1)(n-2).....2(1)(\log x)^0}{(-m)^n .x^{-m}} = \underset{x\to 0+}{\text{Lt}} \frac{\lfloor n\, x^m}{(-m)^n} = 0.$$

Jmp **Example 2.** *Evaluate* $\underset{x\to 0}{Lt}\left(cot^2 x - \frac{1}{x^2}\right)$ *(H.P.U. 1998 ; P.U. 1998 Oct., 2001)*

Pbi: U -2005

Solution. $\underset{x\to 0}{\text{Lt}}\left(\cot^2 x - \frac{1}{x^2}\right)$ (Form $\infty - \infty$)

$$= \underset{x\to 0}{\text{Lt}} \frac{x^2\cos^2 x - \sin^2 x}{x^2\sin^2 x} = \left(\underset{x\to 0}{\text{Lt}} \frac{x^2\cos^2 x - \sin^2 x}{x^4}\right)\left\{\underset{x\to 0}{\text{Lt}}\left(\frac{x}{\sin x}\right)^2\right\}$$

$$= \left(\underset{x\to 0}{\text{Lt}} \frac{x^2(1+\cos 2x - (1-\cos 2x))}{2x^4}\right)(1)^2 = \underset{x\to 0}{\text{Lt}} \frac{x^2 + (x^2+1)\cos 2x - 1}{2x^4}\left(\text{Form}\frac{0}{0}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{2x + (x^2+1)(-2\sin 2x) + (\cos 2x)(2x)}{8x^3}\left(\text{Form}\frac{0}{0}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{2 + (x^2+1)(-4\cos 2x) + (-2\sin 2x)(2x) + (\cos 2x)(2) + (2x)(-2\sin 2x)}{24x^2}$$ (Simplify Num.)

$$= \underset{x\to 0}{\text{Lt}} \frac{2 + (\cos 2x)(-4x^2-2) - 8x\sin 2x}{24x^2}\left(\text{Form}\frac{0}{0}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{(\cos 2x)(-8x) + (-4x^2-2)(-2\sin 2x) - 8x(2\cos 2x) - 8\sin 2x}{48x}$$ (Simplify Num.)

$$= \underset{x\to 0}{\text{Lt}} \frac{-24x\cos 2x + (8x^2-4)\sin 2x}{48}\left(\text{Form}\frac{0}{0}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{(-24)\{x(-2\sin 2x) + \cos 2x\} + (8x^2-4)(2\cos 2x) + (\sin 2x)(16x)}{48}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{(-24)\{0+1\} + (-4)(2) + 0}{48} = -\frac{32}{48} = -\frac{2}{3}$$

Aliter. $\underset{x\to 0}{\text{Lt}}\left(\cot^2 x - \frac{1}{x^2}\right) = \underset{x\to 0}{\text{Lt}}\left(\frac{x^2-\tan^2 x}{x^4}\right)\left(\frac{x}{\tan x}\right)^2 = \underset{x\to 0}{\text{Lt}} \frac{x^2-\tan^2 x}{x^4}\left(\text{Form}\frac{0}{0}\right)$

$$= \underset{x\to 0}{\text{Lt}} \frac{2x - 2\tan x.\sec^2 x}{4x^3} = \underset{x\to 0}{\text{Lt}} \frac{2x - 2\tan x(1+\tan^2 x)}{4x^3}\left(\text{Form}\frac{0}{0}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{2 - 2\sec^2 x - 6\tan^2 x\sec^2 x}{12x^2} = \underset{x\to 0}{\text{Lt}}\left(\frac{-2}{12}\right)\left\{\frac{-1+\sec^2 x + 3\tan^2 x\sec^2 x}{x^2}\right\}$$

$$= \underset{x\to 0}{\text{Lt}}\left(-\frac{1}{6}\right)\left\{\frac{\tan^2 x + 3\tan^2 x\sec^2 x}{x^2}\right\} = \underset{x\to 0}{\text{Lt}}\left(\frac{-1}{6}\right)\left(\frac{\tan x}{x}\right)^2\left\{1+3\sec^2 x\right\}$$

$$= \left(\frac{-1}{6}\right)(1)^2(1+3) = \frac{-4}{6} = \frac{-2}{3}$$

Cor. $\underset{x\to 0}{Lt}\left(\frac{1}{x^2} - \cot^2 x\right) = \frac{2}{3}$ *(K.U. 1997 S)*

Exercise 11(c)

1. Evaluate the following limits :

(i) $\underset{x\to 0+}{\text{Lt}}\ x\log x$ *(G.N.D.U. 1996)* (ii) $\underset{x\to\infty}{\text{Lt}}\ x\tan^{-1}(2/x)$ (iii) $\underset{x\to 0}{\text{Lt}}\ \frac{(e^{1/x})-1}{(e^{1/x})+1}$

[**Hint.** (*iii*) Find R.H.L. and L.H.L. and see that R.H.L. ≠ L.H.L.]

(iv) $\underset{x\to\pi/2-}{\text{Lt}}\left(x-\frac{\pi}{2}\right)(\sec x)$ (v) $\underset{x\to\infty}{\text{Lt}}\ 2^x\sin\left(\frac{a}{2^x}\right)$ (vi) $\underset{x\to\infty}{\text{Lt}}\ x\tan\frac{1}{x}$

(vii) $\underset{x\to 0+}{\text{Lt}}\ x\log\sin x$ (viii) $\underset{x\to\infty}{\text{Lt}}\ x\cot x$ (ix) $\underset{x\to\infty}{\text{Lt}}\left(a^{1/x}-1\right)x$ *(P.U. 1991S)*

2. Evaluate the following limits :

(i) $\underset{x\to 0}{\text{Lt}}\left(\operatorname{cosec} x - \frac{1}{x}\right)$ *(P.U. 1999 S)*

(ii) $\underset{x\to 0}{\text{Lt}}\left(\frac{1}{x} - \cot x\right)$ *(G.N.D.U. 1990, K.U. 1999 S, D.L.U. 2004)*

(iii) $\underset{x\to 0}{\text{Lt}}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$ *(D.L.U. 2004, G.N.D.U. 1996, K.U. 2001, M.D.U. 2002)*

(iv) $\underset{x\to 0}{\text{Lt}}\left(\frac{1}{x^2} - \frac{1}{x\tan x}\right)$ (v) $\underset{x\to 0}{\text{Lt}}\left[\frac{1}{x} - \frac{1}{x^2}\log(1+x)\right]$

(P.U. 1997; G.N.D.U. 2000)

(vi) $\underset{x\to 0}{\text{Lt}}\left(\frac{1}{e^x-1} - \frac{1}{x}\right)$

[**Hint.** Given Limit

$$= \underset{x\to 0}{\text{Lt}}\ \frac{x-e^x+1}{(e^x-1)x} = \underset{x\to 0}{\text{Lt}}\ \frac{1-e^x}{xe^x+(e^x-1)} = \underset{x\to 0}{\text{Lt}}\ \frac{-e^x}{xe^x+e^x+e^x} = \frac{-1}{0+1+1} = \frac{-1}{2}$$

(vii) $\underset{x\to 1}{\text{Lt}}\left(\sec\frac{\pi}{2x}\right)(\log x)$

Answers

1. (*i*) 0 (*ii*) 2 (*iii*) limit does not exist (*iv*) -1 (*v*) a (*vi*) 1 (*vii*) 0 (*viii*) 1 (*ix*) $\log a$

2. (*i*) 0 (*ii*) 0 (*iii*) $-\frac{1}{3}$ (*iv*) $\frac{1}{3}$ (*v*) $\frac{1}{2}$ (*vi*) $-\frac{1}{2}$ (*vii*) $\frac{2}{\pi}$

Art. 7. Form V.

The Indeterminate Forms $\left(0^0, 1^\infty, \infty^0\right)$

To evaluate $\underset{x\to a}{\text{Lt}}\ (f(x))^{g(x)}$ where $\underset{x\to a}{\text{Lt}}\ f(x) = 0, 1$ or ∞ and $\underset{x\to a}{\text{Lt}}\ g(x) = 0, \infty$ or 0

Solution. Let $y = [f(x)]^{g(x)} \Rightarrow \log y = g(x)\log f(x)$

$$\Rightarrow \underset{x\to a}{\text{Lt}}\ \log y = \underset{x\to a}{\text{Lt}}\ [g(x).\log f(x)] = l(\text{say})$$

[Because R.H.S. is of the form $0 . \infty$ and so the limit can be evaluated by earlier methods already discussed]

$$\Rightarrow \log \underset{x\to a}{\text{Lt}}\ y = l \Rightarrow \underset{x\to a}{\text{Lt}}\ y = e^l$$

$$\Rightarrow \underset{x\to a}{\text{Lt}}\ [f(x)^{g(x)} = e^l$$

Solved Examples 11 (d)

Example 1. *Evaluate the following limits :*

(*i*) $\underset{x\to 0+}{Lt}\ x^x$ (*P.U. 1995*) (*ii*) $\underset{x\to 1}{Lt}\ (x)^{1/x-1}$ (*iii*) $\underset{x\to 0+}{Lt}\ (\operatorname{cosec} x)^{1/\log x}$ (*M.D.U. 2000*)

Solution. (*i*) $\underset{x\to 0+}{\text{Lt}}\ x^x$ (Form 0^0)

Let $y = x^x \Rightarrow \log y = \log x^x \Rightarrow \log y \Rightarrow x\log x$

$$\Rightarrow \underset{x\to 0+}{\text{Lt}}\ \log y = \underset{x\to 0+}{\text{Lt}}\ x\log x \text{ (Form } 0.\infty)$$

$$= \underset{x\to 0+}{\text{Lt}}\ \frac{\log x}{1/x}\left(\text{Form}\frac{\infty}{\infty}\right) = \underset{x\to 0+}{\text{Lt}}\ \frac{1/x}{-1/x^2} = \underset{x\to 0+}{\text{Lt}}\ (-x) = 0$$

$$\Rightarrow \underset{x\to 0+}{\text{Lt}}\ y = e^0 = 1 \Rightarrow \underset{x\to 0+}{\text{Lt}}\ x^x = 1$$

(*ii*) $\underset{x\to 1}{\text{Lt}}\ (x)^{\frac{1}{x-1}}$ (Form 1^∞)

Let $y = (x)^{1/x-1} \Rightarrow \log y = \frac{1}{x-1}\log x$

$$\Rightarrow \underset{x\to 1}{\text{Lt}}\ \log y = \underset{x\to 1}{\text{Lt}}\ \frac{\log x}{x-1}\left(\text{Form}\frac{0}{0}\right) = \underset{x\to 1}{\text{Lt}}\ \frac{1/x}{1} = 1$$

$$\Rightarrow \underset{x\to 1}{\text{Lt}}\ y = e^1 \Rightarrow \underset{x\to 1}{\text{Lt}}\ (x)^{1/x-1} = e$$

(*iii*) $\underset{x\to 0+}{\text{Lt}}\ (\operatorname{cosec} x)^{1/\log x}$ (Form ∞^0)

Let $y = (\operatorname{cosec} x)^{1/\log x} \Rightarrow \log y = \dfrac{1}{\log x} \log \operatorname{cosec} x$,

$$\Rightarrow \underset{x\to 0+}{\text{Lt}} \log y = \underset{x\to 0+}{\text{Lt}} \frac{\log \operatorname{cosec} x}{\log x} \left(\text{Form } \frac{\infty}{\infty}\right)$$

$$= \underset{x\to 0+}{\text{Lt}} \frac{\left(\dfrac{1}{(\operatorname{cosec} x)}\right)(-\operatorname{cosec} x \cot x)}{\dfrac{1}{x}} = \underset{x\to 0+}{\text{Lt}} \left(-\frac{x}{\tan x}\right) = -1$$

$$\Rightarrow \underset{x\to 0+}{\text{Lt}}\ y = e^{-1} \Rightarrow \underset{x\to 0+}{\text{Lt}} (\operatorname{cosec} x)^{1/\log x} = e^{-1} = \frac{1}{e}$$

Example 2. *Evaluate* $\underset{x\to 0}{\text{Lt}} \left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$ Pbi-U-2009

(*K.U. 1997 S, P.U. 2002, 2001 S, M.D.U. 1999*)

Solution. $\underset{x\to 0}{\text{Lt}} \left(\dfrac{\sin x}{x}\right)^{1/x^2}$ $[\text{Form } 1^\infty]$

Let $y = \left(\dfrac{\sin x}{x}\right)^{1/x^2} \Rightarrow \log y = \dfrac{1}{x^2} \log\left(\dfrac{\sin x}{x}\right)$

$$\Rightarrow \underset{x\to 0}{\text{Lt}} \log y = \underset{x\to 0}{\text{Lt}} \frac{\log\left(\dfrac{\sin x}{x}\right)}{x^2} \left(\text{Form } \frac{0}{0}\right) = \underset{x\to 0}{\text{Lt}} \frac{\left(\dfrac{\sin x}{x}\right)^{-1} \left\{\dfrac{x\cos x - \sin x}{x^2}\right\}}{2x}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{x\cos x - \sin x}{2x^3} \left(\text{Form } \frac{0}{0}\right) = \underset{x\to 0}{\text{Lt}} \frac{x(-\sin x) + \cos x - \cos x}{6x^2}$$

$$= \underset{x\to 0}{\text{Lt}} \left(-\frac{1}{6}\right)\left(\frac{\sin x}{x}\right) = \left(-\frac{1}{6}\right)(1) = -\frac{1}{6}$$

$$\Rightarrow \underset{x\to 0}{\text{Lt}}\ y = e^{-1/6} \Rightarrow \underset{x\to 0}{\text{Lt}} \left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$$

Exercise 11(d)

Evaluate the following limits :

1. (*i*) $\underset{x\to 0}{\text{Lt}} (\cosh x)^{1/x}$ (*ii*) $\underset{x\to 0+}{\text{Lt}} (1+x)^{1/x}$

   (*iii*) $\underset{x\to 0}{\text{Lt}} (\cos x)^{\cot^2 x}$ (*iv*) $\underset{x\to \pi/2}{\text{Lt}} (\sin x)^{\tan x}$ (*H.P.U. 1998, 1997*)

2. (*i*) $\underset{x\to \infty}{\text{Lt}} (1+x)^{1/x}$ (*ii*) $\underset{x\to 0+}{\text{Lt}} (\cot x)^x$ (*P.U. 1999 S; H.P.U. 1999*)

3. (*i*) $\underset{x\to 0+}{\text{Lt}} \dfrac{x^x - 1}{x}$ Imp (*ii*) $\underset{x\to 0+}{\text{Lt}} (\sin x)^{\tan x}$ (*H.P.U. 1998; P.U. 1995*)

   (*iii*) $\underset{x\to 0+}{\text{Lt}} (x)^{1/\log x}$ (*iv*) $\underset{x\to 0+}{\text{Lt}} (x)^{\sin x}$

(v) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{a^x+b^x}{2}\right)^{1/x}$ (M.D.U. 2004, 1998)   (vi) $\underset{x\to \infty}{\text{Lt}}\left(\dfrac{\pi}{2}-\tan^{-1}x\right)^{1/x}$ (K.U. 1998 S, K.U. 2004)

4. (i) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{\tan x}{x}\right)^{1/x}$ (ii) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{\tan x}{x}\right)^{1/x^2}$ April 2009 (M.D.U. 1996; P.U. 2000 S; H.P.U. 2003)

(iii) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{\sinh x}{x}\right)^{1/x^2}$ (P.U. 2003 ; K.U. 1995)   (iv) $\underset{x\to 0}{\text{Lt}}\left[\dfrac{2(\cosh x-1)}{x^2}\right]^{1/x^2}$ (P.U. 1991 S)

(v) $\underset{x\to \frac{\pi}{4}}{\text{Lt}}(\tan x)^{\frac{1}{x-\pi/4}}$ (P.U. 1994)   (vi) $\underset{x\to \frac{\pi}{4}}{\text{Lt}}(\tan x)^{\tan 2x}$ (G.N.D.U. 1999 April)

(vii) $\underset{x\to 1}{\text{Lt}}(1-x^2)^{\frac{1}{\log(1-x)}}$ (viii) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{\sin x}{x}\right)^{1/x}$ (P.U. 1999)

Answers

1. (i) 1 (ii) e (iii) $e^{-1/2}$ (iv) 1   2. (i) 1 (ii) 1

3. (i) $-\infty$ (ii) 1 (iii) e (iv) 1 (v) $\sqrt{ab}$ (vi) 1

4. (i) 1 (ii) $e^{1/3}$ (iii) $e^{1/6}$ (iv) $e^{1/12}$ (v) e^2 (vi) $(1/e)$ (vii) e (viii) 1

USE OF EXPANSION IN EVALUATION OF LIMITS

Solved Examples 11 (e)

Example 1. *Evaluate* $\underset{x\to 0}{Lt}\dfrac{(1+x)^{\frac{1}{x}}-e+\dfrac{ex}{2}}{x^2}$

(H.P.U. 1996 ; G.N.D.U. 1991 S; Pbi.U. 2003 ; K.U. 1993)

Solution We know that $\underset{x\to 0}{\text{Lt}}(1+x)^{1/x}=e$ ...(1)

and $e^x = 1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+.....$ ...(2)

and $\log(1+x) = x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+..........$ ...(3)

To Expand $(1+x)^{1/x}$

Let $y = (1+x)^{1/x}$

$\Rightarrow$ $\log y = \dfrac{1}{x}\log(1+x)=\dfrac{1}{x}\left(x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+.......\right)=\left[1-\dfrac{x}{2}+\dfrac{x^2}{3}-...\right]$

$$y = e^{\left[1-\frac{x}{2}+\frac{x^2}{3}\cdots\right]} \Rightarrow y = e^1 e^{\left[-\frac{x}{2}+\frac{x^2}{3}\cdots\right]}$$

$$\Rightarrow \quad y = e\left[1+\left[-\frac{x}{2}+\frac{x^2}{3}-\ldots\right]+\frac{1}{\underline{|2}}\left[-\frac{x}{2}+\frac{x^2}{3}-\ldots\right]^2+\ldots\right]$$

$$\Rightarrow \quad y = e\left[1-\frac{x}{2}+\frac{11}{24}x^2-\ldots\right]$$

Hence, $$\underset{x\to 0}{\text{Lt}} \frac{(1+x)^{1/x}-e+\frac{1}{2}ex}{x^2} = \underset{x\to 0}{\text{Lt}} \frac{e\left[1-\frac{x}{2}+\frac{11}{24}x^2-\ldots\right]-e+\frac{1}{2}ex}{x^2}$$ $-\frac{11ex^2}{24}$ / x^3

$$= \underset{x\to 0}{\text{Lt}}\left[\frac{11e}{24}+\text{terms containing } x\right] = \frac{11e}{24}$$

Exercise 11(e)

1. Evaluate the limit $\underset{x\to 0}{\text{Lt}} \frac{(1+x)^{\frac{1}{x}}-e}{x}$ (K.U. 1996; P.U. 1994)

2. Evaluate the limit $\underset{x\to 0}{\text{Lt}} \frac{(1+x)^{\frac{1}{x}}-e+\frac{ex}{2}-\frac{11ex^2}{24}}{x^3}$ (H.P.U. 1999)

Answers

1. $\frac{-e}{2}$ 2. $\frac{-7e}{16}$

Miscellaneous Exercise

1. Evaluate the following limits

(i) $\underset{x\to 0}{\text{Lt}} \frac{(1+x)^n-1}{x}$ (ii) $\underset{x\to 0}{\text{Lt}} \frac{(1+x)^a-ax-1}{x^2}; (a>1)$

(iii) $\underset{x\to 0}{\text{Lt}} \frac{1+\sin x-\cos x+\log(1-x)}{x\tan^2 x}$ (M.D.U. 1995; P.U. 1999)

(iv) $\underset{x\to\infty}{\text{Lt}} \frac{\sin\frac{1}{x}}{\tan^{-1}\frac{1}{x}}$ (v) $\underset{x\to 0}{\text{Lt}} \log_x \sin x$

2. Evaluate the following limits

(i) $\underset{x\to 1}{\text{Lt}} \sec\frac{\pi x}{2}\log\frac{1}{x}$ (ii) $\underset{x\to 0}{\text{Lt}}\left[\frac{1}{2x}-\frac{1}{x(e^{\pi x}+1)}\right]$

Evaluati $\lim_{x\to 0}\left[\tan\left(\frac{\pi}{4}+x\right)\right]^{\frac{1}{x}}$ Ans e^2

(iii) $\underset{x\to 0}{\text{Lt}}\left\{(1+x)(x^2e^{1/x^2})\left(\frac{e^x-e^{-x}}{\log_e^{(1+x)}}\right)\right\}$

3. Evaluate the following limits

(i) $\underset{x\to c}{\text{Lt}}\left(2-\frac{x}{c}\right)^{\tan\frac{\pi x}{2c}}$ s(ii) $\underset{x\to a}{\text{Lt}}(x-a)^{x-a}$

(iii) $\underset{x\to 0}{\text{Lt}}(1+2x)^{\frac{x+5}{x}}$ (iv) $\underset{x\to 0}{\text{Lt}}(e^x+4x)^{\frac{1}{x}}$

(v) $\underset{x\to 0}{\text{Lt}}\left(\frac{\sin x}{x}\right)^{1/x}$ (viii) *(P.U. 1999, 1997, 1996 S, 1995 S, 1994 S)*

4. (i) Prove that $\underset{x\to\infty}{\text{Lt}}\, x\left[\left(1+\frac{a}{x}\right)^x-e^a\right]=\frac{-e^aa^2}{2}$ *(P.U. 1991)*

(ii) Prove that $\underset{x\to\infty}{\text{Lt}}\, x\left[\left(1-\frac{a}{x}\right)^{-x}-\left(1+\frac{a}{x}\right)^x\right]=a^2e^a$

(iii) Prove that $\underset{x\to 0}{\text{Lt}}\left(\frac{\sin x-e^x+1}{x}\right)\cos\left(\frac{1}{x}\right)=0$ *(P.U. 1994)*

(iv) Prove that $\underset{x\to 0}{\text{Lt}}\left(\frac{\sin x-e^x+1}{x}\right)\left(\sin\frac{1}{x}\right)=0$ *(H.P.U. 2003)*

(v) Prove that $\underset{x\to 0}{\text{Lt}}\left(\frac{e^{\sin x}-1-\tan x}{x}\right)\cos\left(\frac{1}{x}\right)=0$ *(P.U. 1997)*

(vi) Prove that $\underset{x\to\frac{\pi}{2}}{\text{Lt}}(1-\sin x)\tan x=0$ *(Pbi.U. 1994)*

(vii) Prove that $\underset{x\to 0}{\text{Lt}}\frac{x^3e^{x^4/4}-(\sin x^2)^{3/2}}{x^7}=\frac{1}{2}$

Hint. $e^{x^4/4}=1+\frac{x^4}{4}+\frac{1}{\underline{|2}}\left(\frac{x^4}{4}\right)^2+\text{..........}=1+\frac{x^4}{4}+\frac{x^8}{32}+\text{.......}$

and $(\sin x^2)^{3/2}=\left[x^2-\frac{(x^2)^3}{\underline{|3}}+\frac{(x^2)^5}{\underline{|5}}-\text{........}\right]^{3/2}=x^3\left[1-\frac{x^4}{6}+\text{.......}\right]^{3/2}$

$=x^3\left[1-\frac{x^4}{4}+\text{.........}\right]$ (Using Binomial Theorem)

(viii) Prove that $\underset{x\to 0}{\text{Lt}}\left(\frac{a_1^x+a_2^x+\text{.......}+a_n^x}{n}\right)^{1/x}=(a_1a_2\text{......}a_n)^{1/n}$

(ix) Prove that $\underset{x\to b}{\text{Lt}}\frac{x^b-b^x}{x^x-b^b}=\frac{1-\log b}{1+\log b}$ *(M.D.U. 1999)*

□□□